THE FUNDAMENTAL GAP CONJECTURE FOR POLYGONAL DOMAINS

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Abstract. In 1985, S. T. Yau made the following "fundamental gap conjecture," [25]. *For a convex domain* $\Omega \subset \mathbb{R}^n$,

$$\xi(\Omega) := d^2 \left(\lambda_2(\Omega) - \lambda_1(\Omega)\right) \geq 3\pi^2 \tag{0.1}$$

where d *is the diameter of the domain, and* $0 < \lambda_1(\Omega) < \lambda_2(\Omega)$ *are the first two eigenvalues of the Euclidean Laplacian on* Ω *with Dirichlet boundary condition.* The scalar invariant ξ is the *gap function.* We restrict attention to planar domains. Our main result is a compactness theorem for the gap function when the domain is a triangle in $\mathbb{R}^2$. This result shows that for any triangles which collapse to the unit interval, the gap function is unbounded. Due to numerical methods, we expect that the fundamental gap conjecture holds for all triangular domains in $\mathbb{R}^2$. We show with examples that the behavior of the gap for collapsing polygonal domains is quite delicate. These examples motivate a technical result for collapsing polygonal domains giving conditions under which the gap function either remains bounded or becomes infinite. Our work initiates a general program to prove the fundamental gap conjecture using convex polygonal domains.

1. Motivation and results

For a Schrödinger operator on a compact convex domain, after the first eigenvalue the next natural object to study is the gap between the first two eigenvalues, known as the *fundamental gap.* This includes the work of [29], [16], [10], [2], [25], [27], [11], and many other authors. While it is always interesting to understand the interaction between the eigenvalues of a differential operator and the geometry of the domain, beyond purely mathematical implications the fundamental gap has physical implications. For the heat equation, the gap controls the rate of collapse of any initial state toward a state dominated by the first eigenvalue and is of central interest in statistical mechanics and quantum field theory. In analysis, the gap is important to refinements of the Poincaré inequality and à priori estimates. Numerically, the gap can be used to control the rate of convergence of numerical computation methods such as discretization or finite element method by which one uses matrices to approximate a differential operator. The ability to solve for the first eigenvalue and eigenvector of these matrices is controlled by the size of the gap between the first eigenvalue and the rest of the spectrum. Understanding the behavior of the gap for collapsing convex polygonal domains is also relevant to computer graphics image rendering [9].

Key words and phrases. spectral geometry, polygonal domain, triangle, sector, eigenvalues, Dirichlet Laplacian, fundamental gap.

In [25], using gradient estimates, the lower bound $\pi^2/4$ was proven for the gap function. In [29], further generalizations of the above result to Schrödinger operators were obtained. More importantly, in that paper, the case when the potential function is not convex was discussed and the lower bound of the gap was obtained. After the Hessian of the log of the first eigenfunction was estimated, the result is the counterpart of the Li-Yau estimate [16] on the first eigenvalue with Ricci curvature being bounded below. Note that studying the Schrödinger operators rather than the Laplacian is not only a generalization but is also necessary. For example, a simple proof of the log concavity of the first eigenfunction (A theorem of Brascamp-Lieb) was obtained in [25] by viewing the Laplacian as the limiting operator of a series of Schrödinger operators. The gap estimate was sharpened in [29] and [31]; currently the best lower bound for the gap function on the Dirichlet Laplacian on a convex domain in $\mathbb{R}^n$ is π^2. A sharp upper bound for the gap function on convex domains in $\mathbb{R}^n$ is given in Proposition 2 of [11].

Our first result is a compactness theorem for the fundamental gap on triangular domains. Since ξ is invariant under scaling of the domain, we consider ξ on the moduli space of triangles which consists of all similarity classes of triangles. In terms of the smallest two angles, the moduli space of triangles may be represented by a triangle with one side removed in the $\alpha \times \beta$ plane, where a point (α, β) represents the similarity class of triangles with angles $\alpha\pi \leq \beta\pi \leq \pi - \alpha\pi - \beta\pi$; see Figure 1.

Theorem 1. *Let P be the set of all similarity classes of triangles. Then, $\xi : P \to \mathbb{R}$ as defined in (0.1) is a proper map:*

$$P_C := \{T \in P \mid \xi(T) \leq C\}$$

is a compact set.

This theorem shows that there exists a minimizer for the fundamental gap on triangular domains, and in particular, we expect the following.

Conjecture 1. *Let $\xi : P \to \mathbb{R}$ be defined by (0.1). Then, ξ has no extrema on the interior of the moduli space of triangles, and moreover, ξ is monotonic on the non-degenerate boundaries of the moduli space of triangles.*

The non-degenerate boundaries of the moduli space of triangles consist of isosceles triangles, and the equilateral triangle lies at the non-degenerate vertex. Conjecture 1 states that ξ has neither maxima nor minima for scalene non-degenerate triangles and moreover, for isosceles triangles with angles $\alpha\pi = \alpha\pi < \pi - 2\alpha\pi$ or angles $\alpha\pi < \pi/2 - \alpha\pi/2 = \pi/2 - \alpha\pi/2$, $\xi(\alpha)$ is strictly decreasing on $\alpha \in (0, 1/3)$. The eigenvalues and eigenfunctions of the equilateral triangle T are explicitly computable [13], and

$$\xi(T) = \frac{64\pi^2}{9}.$$

Consequently, Conjecture 1 would immediately imply the following conjecture, which also appeared in [2].

Conjecture 2. *Let $T \subset \mathbb{R}^2$ be a triangular domain. Then*

$$\xi(T) \geq \frac{64\pi^2}{9},$$

where equality holds iff T is equilateral.

Our second theorem is a technical generalization of the compactness theorem which provides conditions under which the fundamental gap remains bounded or becomes infinite as convex polygonal domains collapse to the unit interval.

Theorem 2. *Let $\{Q_n\}_{n\in\mathbb{N}}$ be convex m-gons so that $\lim_{n\to\infty} Q_n = [0,1]$. Assume $Q_n \subset \{(x,y)\in\mathbb{R}^2 : x, y \geq 0\}$, and that the longest side S_n of Q_n lies on the x-axis. Let the height h_n of Q_n be defined by*

$$h_n := \inf\{b : Q_n \subset [0,a]\times[0,b]\}. \tag{1.1}$$

(1) *If there exist rectangles $R_n \supset Q_n \supset r_n$ so that*

$$Area(R_n) - Area(r_n) \leq O(h_n^3),$$

then $\xi(Q_n)$ is bounded as $n\to\infty$.

(2) *If there exists a convex inscribed polygon $U_n \subset Q_n$ for which the following conditions are satisfied, then $\xi(Q_n)\to\infty$ as $n\to\infty$.*
   (a) *The diameter of $U_n \to 0$ as $n\to\infty$.*
   (b) *One side Σ_n of U_n is contained in S_n.*
   (c) *The height of $U_n = h_n$.*
   (d) *The height of $V_n := Q_n - U_n$, satisfies $h(V_n) \leq h_n - O(h_n^x)$ for some $x < \frac{5}{3}$.*

In section 2, we study the behavior of the fundamental gap on triangular domains and prove the compactness theorem. In section 3, we provide examples of collapsing quadrilateral domains and prove our theorem for polygonal domains. Concluding remarks comprise section 4.

2. The fundamental gap on triangular domains

We consider here the Euclidean Laplacian on $\mathbb{R}^2$, which in rectangular and polar coordinates is respectively,

$$\Delta = -(\partial_x^2 + \partial_y^2), \quad \Delta = -\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2.$$

The Laplacian on a polygonal domain in $\mathbb{R}^2$ with Dirichlet boundary condition has discrete spectrum

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots .$$

The vertex of a circular sector with the appropriate angle is a good local model for the corresponding vertex of a triangle [5]. This motivates one to approximate the eigenvalues of triangles using the variational principle and the eigenvalues for a circular sector, which can be explicitly computed using separation of variables for the Laplacian in polar coordinates.

2.1. Eigenvalues for circular sectors.

The eigenvalues of a circular sector of opening angle $\alpha\pi$ and radius 1, with Dirichlet boundary condition are

$$\{\lambda_{k,s}\}_{k,s\in\mathbb{N}} = \{(j_{k/\alpha,s})^2\}_{k,s\in\mathbb{N}}, \tag{2.1}$$

where $j_{\nu,s}$ is the s^{th} zero of the Bessel function J_ν of order ν. Note that if the radius of the sector is r, the eigenvalues scale by r^{-2}.

We recall the variational or mini-max principle for the first eigenvalue,

$$\lambda_1 = \inf_{f\in \mathrm{Dom}(\Delta): f\neq 0} \frac{\int_T |\nabla f|^2 dxdy}{\int_T f^2 dxdy}, \tag{2.2}$$

where $\mathrm{Dom}(\Delta)$ is the domain of Δ, so the infimum is taken over functions satisfying the boundary condition. This is the infimum of the Rayleigh-Ritz quotient. Similarly, the second eigenvalue is

$$\lambda_2 = \inf_{f \in \mathrm{Dom}(\Delta): f \neq 0, f \perp f_1} \frac{\int_T |\nabla f|^2 dxdy}{\int_T f^2 dxdy}, \tag{2.3}$$

where f_1 is the eigenfunction for λ_1, and orthogonality is with respect to $\mathcal{L}^2$. These formulae are in [4] and [6]. A further property of the eigenvalues proven in [4] and [6] is domain monotonicity: if $\Omega \subset \Omega'$ then for the Dirichlet eigenvalues

$$\lambda_k(\Omega) \geq \lambda_k(\Omega').$$

2.1.1. *Asymptotic formulae for zeros of Bessel functions and eigenvalues of circular sectors.* The following formulae are due to [14], [18], and [24] for Bessel functions of real order and date back to the work of [28] and [21] for Bessel functions of integer order. The first and second zeros of the Bessel function of order ν are

$$j_{\nu,i} = \nu - \frac{a_i}{2^{1/3}} \nu^{1/3} + O(\nu^{-1/3}), \quad i = 1, 2, \tag{2.4}$$

where a_i is the i^{th} zero of the Airy function of the first kind so that

$$a_1 \approx -2.33811, \text{ and } a_2 \approx -4.08795. \tag{2.5}$$

Consequently, we have the following estimates for the first two Dirichlet eigenvalues of the circular sector of opening angle $\alpha\pi$ and radius one

$$\lambda_i(\alpha\pi) = \frac{1}{\alpha^2} + \frac{c_i}{\alpha^{4/3}} + O(\alpha^{-1}), \quad i = 1, 2, \tag{2.6}$$

where

$$c_1 = -a_1 2^{2/3} \approx 3.71151827 \text{ and } c_2 = -a_2 2^{2/3} \approx 6.48921613. \tag{2.7}$$

We will also use the constant[1]

$$c_1' = -a_1' 2^{2/3} \approx 1.61722832. \tag{2.8}$$

2.2. **Gap behavior approaching the degenerate boundary of the moduli space of triangles.** Since the gap function is invariant under scaling, we restrict to triangles with diameter one. Such a triangle has angles

$$0 < \alpha\pi \leq \beta\pi \leq \pi - \alpha\pi - \beta\pi.$$

The moduli space of all such triangles is itself a triangle in the $\alpha \times \beta$ plane; see Figure 1. We are interested in the behavior of ξ on this moduli space, and in particular the behavior of ξ approaching the degenerate boundary of P which is the dashed vertical segment in Figure 1.

[1]This constant arises from the asymptotic formula for the first zero of the derivative of the Bessel function which is related to the first Dirichlet eigenvalue of an obtuse isosceles triangle; see [10].

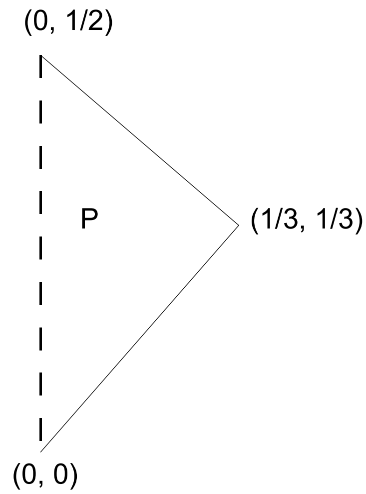

FIGURE 1. Moduli space of triangles.

2.2.1. *Gap behavior away from the origin in the moduli space of triangles.* Away from the origin in the moduli space of triangles, we are able to prove that $\xi \sim \alpha^{-4/3}$ as $\alpha \to 0$. Consider the triangle T with angles $0 < \alpha\pi \leq \beta\pi \leq \pi - \alpha\pi - \beta\pi$, and assume for some fixed $\epsilon > 0$, $\beta \geq \epsilon$. Let the side opposite $\alpha\pi$ have length A, the side opposite $\beta\pi$ have length B, and the third side have length one. The law of sines states that

$$\frac{\sin(\alpha\pi)}{A} = \frac{\sin(\beta\pi)}{B} = \frac{\sin(\pi - \alpha\pi - \beta\pi)}{1}.$$

Note that

$$B = \frac{\sin(\beta\pi)}{\sin(\alpha\pi + \beta\pi)} = 1 - O(\alpha) \text{ as } \alpha \to 0.$$

For α small, we may approximate the eigenvalues using two sectors, a larger sector of radius 1, and a smaller sector of radius $1 - O(\alpha)$, both with opening angle $\alpha\pi$; see Figure 2. By domain monotonicity,

$$\lambda_2(T) \geq \lambda_2(\text{large sector}) = \lambda_2(S),$$
$$\lambda_1(T) \leq \lambda_1(\text{small sector}) = \lambda_1(s).$$

Then,

$$\xi(T) \geq \lambda_2(S) - \lambda_1(s) = \frac{1}{\alpha^2} + \frac{c_2}{\alpha^{4/3}} - \frac{1}{\alpha^2(1-O(\alpha))^2} - \frac{c_1}{\alpha^{4/3}(1-O(\alpha))^2} + O(\alpha^{-1}).$$

This shows that $\xi(T)$ behaves like $\alpha^{-4/3}$ as $\alpha \to 0$.

For some specific trajectories approaching the origin in the moduli space, we are also able to show that $\xi \sim \alpha^{-4/3}$ as $\alpha \to 0$. If $\alpha, \beta \to 0$ so that $\alpha = o(\beta^3)$, the following estimates demonstrate that $\xi \sim \alpha^{-4/3}$ as $\alpha \to 0$. Consider the triangle with smallest angle $\alpha\pi$ and vertices A, B, E. When $x = |DE|$ is very small we approximate $\xi(T)$ using the larger triangle PBE, which is similar to the triangle ABD; see Figure 3. By domain monotonicity,

$$\lambda_2(ABE) > \lambda_2(PBE), \text{ and } \lambda_1(ABE) < \lambda_1(ABD),$$

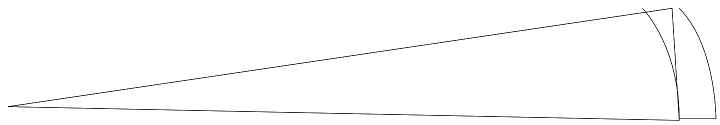

FIGURE 2. Triangle with one collapsing angle and approximating circular sectors.

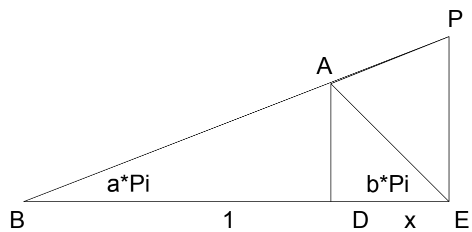

FIGURE 3. Collapsing triangle and approximating triangle.

so that

$$\lambda_2(ABE) - \lambda_1(ABE) > \lambda_2(PBE) - \lambda_1(ABD) = \lambda_2(ABD)(1+x)^{-2} - \lambda_1(ABD).$$

Since

$$\lambda_2(ABD) = \frac{1}{\alpha^2} + \frac{c_2}{\alpha^{4/3}} + O(\alpha^{-1}), \text{ and } \lambda_1(ABD) = \frac{1}{\alpha^2} + \frac{c_1}{\alpha^{4/3}} + O(\alpha^{-1}),$$

for $x < 1$ we expand

$$\lambda_2(PBE) = \frac{1}{\alpha^2} - \frac{2x}{\alpha^2} + \frac{c_2}{\alpha^{4/3}} - \frac{2c_2 x}{\alpha^{4/3}} + O(\alpha^{-1}).$$

For

$$x < \frac{c_2\alpha^{2/3}}{2},$$

we then have the estimate

$$\lambda_2(ABE) - \lambda_1(ABE) \geq \frac{\delta}{\alpha^{4/3}},$$

for some $\delta > 0$.

2.2.2. *Gap behavior approaching the origin along a non-degenerate boundary in the moduli space.* If we approach the origin in the moduli space along trajectories near the non-degenerate boundary which meets the origin, these triangles are "almost" isosceles, and we can show that $\xi \sim \alpha^{-4/3}$ as $\alpha \to 0$. First, we consider obtuse isosceles triangles with diameter one. By [10] Theorem 5.3, the first eigenvalue is

$$\lambda_1(T) = \frac{4}{\alpha^2} + \frac{4c_1'}{\alpha^{4/3}} + O(\alpha^{-2/3}).$$

The second eigenvalue of T was also computed in [10],

$$\lambda_2(T) = \frac{4}{\alpha^2} + \frac{4c_1}{\alpha^{4/3}} + O(\alpha^{-2/3}).$$

Then,

$$\xi(T) \geq \frac{4(c_1 - c_1')}{\alpha^{4/3}} + O(\alpha^{-2/3}),$$

which shows that $\xi(T) \sim \alpha^{-4/3}$ as $\alpha \to 0$. Note that [10] computed further terms in the asymptotic expansion of ξ in this case.

When two angles are collapsing at approximately the same rate so that the triangle is "almost" isosceles, we estimate using the variational principle. Let the two collapsing angles be $\alpha\pi$ and $\beta\pi$. Assume that there exists a constant $c \in (0, 1)$, to be specified at the end of this argument, so that

$$\alpha \leq \beta, \quad c \leq \frac{\alpha}{\beta} \leq 1 \text{ as } \alpha, \beta \to 0.$$

Let the third angle of the triangle be γ with opposite (longest) side length 1, and let A and B be the sides opposite angles $\alpha\pi$ and $\beta\pi$, respectively. In Figure 4, this is triangle PRT. Consider the isosceles triangle with angles $\beta\pi, \beta\pi, \pi - 2\beta\pi$, contained in the original triangle; in Figure 4 this is triangle QRT. The sides of this triangle are $A, A, 2A\cos(\beta\pi)$. Let $\lambda_1(\beta\pi)$ be the first Dirichlet eigenvalue for this triangle and let λ_1 be the first eigenvalue of the original triangle. By domain monotonicity,

$$\lambda_1 \leq \lambda_1(\beta\pi) = \frac{1}{(A\cos(\beta\pi))^2\beta^2} + \frac{c_1'}{(A\cos(\beta\pi))^2\beta^{4/3}} + O(\beta^{-2/3}).$$

By domain monotonicity, we approximate λ_2 from below by the second eigenvalue of the isosceles triangle with angles $\alpha\pi, \alpha\pi, \pi - 2\alpha\pi$ and sides $B, B, 2B\cos(\alpha\pi)$, which contains the original triangle and is triangle PST in Figure 4. The second eigenvalue for this isosceles triangle as computed by [10] is

$$\lambda_2(\alpha\pi) = \frac{1}{(B\cos(\alpha\pi))^2\alpha^2} + \frac{c_1}{(B\cos(\alpha\pi))^2\alpha^{4/3}} + O(\alpha^{-2/3}).$$

Therefore,

$$\lambda_2 - \lambda_1 \geq \frac{1}{(B\cos(\alpha\pi))^2\alpha^2} + \frac{c_1}{(B\cos(\alpha\pi))^2\alpha^{4/3}} - \frac{1}{(A\cos(\beta\pi))^2\beta^2} - \frac{c_1'}{(A\cos(\beta\pi))^2\beta^{4/3}} + O(\beta^{-2/3}).$$

By the law of sines,

$$A\sin(\beta\pi) = B\sin(\alpha\pi) \Rightarrow A\beta = B\alpha + O(\beta^3),$$

when α and β are small. Moreover, $\cos(\alpha\pi) = 1 + O(\alpha^2)$, and $\cos(\beta\pi) = 1 + O(\beta^2)$ as $\alpha, \beta \to 0$. This shows that we need to approximate

$$\frac{c_1}{B^2\alpha^{4/3}} - \frac{c_1'}{A^2\beta^{4/3}},$$

By the hypothesis,

$$c\beta \leq \alpha \leq \beta,$$

so that when α and β are small, the law of sines gives

$$cB \leq A \leq B.$$

Then,

$$\lambda_2 - \lambda_1 \geq \frac{c^2 c_1 \beta^{4/3} - c_1' \alpha^{4/3}}{B^2 c^2 \alpha^{4/3}\beta^{4/3}} + O(\beta^{-2/3}).$$

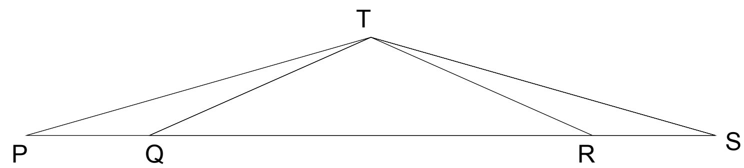

FIGURE 4. Triangle with two angles collapsing and approximating isosceles triangles.

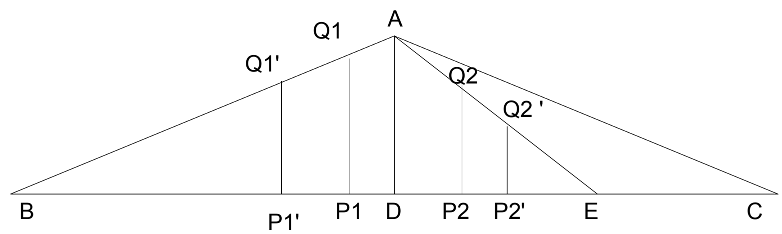

FIGURE 5. Arbitrary collapsing triangle.

So, we require

$$c_1' \alpha^{4/3} < c^2 c_1 \beta^{4/3}.$$

Since $c \le \frac{\alpha}{\beta} \le 1$, this will be satisfied for

$$c > \left(\frac{c_1'}{c_1} \right)^{1/2} \approx 0.660100001.$$

This shows that for collapsing triangles with $c < \frac{\alpha}{\beta}$, $\xi(T)$ is bounded below by a constant multiple of $\alpha^{-4/3}$.

Due to these calculations and the work of [3], we expect that ξ is a polyhomogeneous function on the moduli space of triangles blown up at the origin. Such a regularity result for the gap function is an interesting open problem for polygonal domains, and in particular, it would show that for any family of collapsing triangles with smallest angle α, $\xi \sim \alpha^{-4/3}$ as the triangles collapse.

2.3. **Proof of the compactness theorem.** Assume the smallest angle of the triangle $\angle B = \alpha\pi$; see Figure 5. We wish to estimate $\lambda_2(ABE) - \lambda_1(ABE)$ from below. Assume $|DE| \le |BD| = 1$, and fix

$$0 < \epsilon < \frac{2}{9}.$$

Let P_1 be between B and D so that $|P_1 D| = \alpha^\epsilon$, and P_1' be between B and P_1 so that $|P_1' D| = 2\alpha^\epsilon$. If $|DE| > 2\alpha^\epsilon$, let P_2 be between D and E so that $|P_2 D| = \alpha^\epsilon$, and P_2' be between P_2 and E so that $|P_2' D| = 2\alpha^\epsilon$. If $|DE| \le 2\alpha^\epsilon$, we do not define or use the points P_2, P_2'. If $|DE| > 2\alpha^\epsilon$ let U be the trapezoid $AQ_1P_1P_2Q_2$ and similarly let $U' = AQ_1'P_1'P_2'Q_2'$. Let $V = ABE - U$ and let $V' = ABE - U'$; see Figure 5. If $|DE| \le 2\alpha^\epsilon$, we let $U = AQ_1P_1E$, $U' = AQ_1'P_1'E$, $V = ABE - U$

and $V' = ABE - U'$. In the estimates to follow, we show that we may estimate $\lambda_2(ABE) - \lambda_1(ABE)$ using $\lambda_2(U') - \lambda_1(U')$.

Let f_i be the eigenfunction for $\lambda_i = \lambda_i(ABE)$, $i = 1, 2$. The height of V is at most

$$(1 - \alpha^\epsilon)\tan(\alpha\pi) \approx (1 - \alpha^\epsilon)\pi\alpha.$$

By the one dimensional Poincaré Inequality

$$\frac{\int_V |\nabla f_i|^2}{\int_V f_i^2} \geq \frac{\pi^2}{(1-\alpha^\epsilon)^2\pi^2\alpha^2}, \quad \frac{\int_U |\nabla f_i|^2}{\int_U f_i^2} \geq \frac{\pi^2}{\pi^2\alpha^2}, \text{ and } \frac{\int_{U'} |\nabla f_i|^2}{\int_{U'} f_i^2} \geq \frac{\pi^2}{\pi^2\alpha^2}.$$

Assume

$$\int_{ABE} f_i^2 = 1, \text{ and } \int_V f_i^2 = \beta.$$

By the variational principle,

$$\frac{\beta}{(1-\alpha^\epsilon)^2\alpha^2} + \frac{1-\beta}{\alpha^2} \leq \int_V |\nabla f_i|^2 + \int_U |\nabla f_i|^2 = \lambda_i \leq \lambda_2(ABD) = \frac{1}{\alpha^2} + \frac{c_2}{\alpha^{4/3}} + O(\alpha^{-1}).$$

Therefore,

$$\beta \leq \frac{\alpha^{2/3} c_2 (1-\alpha^\epsilon)^2}{\alpha^\epsilon(2-\alpha^\epsilon)} \leq \alpha^{2/3-\epsilon}.$$

For simplicity in the arguments to follow, we replace all constant factors multiplying positive powers of α by a constant factor of 1, since no generality is lost as $\alpha \to 0$.

Let ρ be a smooth compactly supported function so that

$$\rho|_U \equiv 1, \quad \rho|_{V'} \equiv 0. \tag{2.9}$$

We may choose ρ so that

$$|\nabla\rho| \leq \frac{1}{\alpha^\epsilon}, \quad \text{and} \quad |\Delta\rho|, |\Delta(\rho^2)| \leq \frac{1}{\alpha^{2\epsilon}}. \tag{2.10}$$

For the arguments to follow, we use the sign convention for the Euclidean Laplacian so that $-\Delta$ has positive spectrum. Note that

$$-(\rho f_i)\Delta(\rho f_i) = \lambda_i \rho^2 f_i^2 - f_i^2 \rho\Delta\rho - 2 f_i \rho(\nabla\rho)(\nabla f_i). \tag{2.11}$$

2.3.1. *Estimate for $\lambda_1(U')$.* Since ρ vanishes on the boundary of U', ρf_1 is an admissible test function for the Rayleigh quotient on U' (2.2) which we may use to estimate $\lambda_1(U')$ from above. By (2.11),

$$\lambda_1(U') \leq \lambda_1 + \frac{\int_{U'} \left(-\rho\Delta\rho f_1^2 - 2\rho\nabla\rho f_1 \nabla f_1\right)}{\int_{U'} \rho^2 f^2}.$$

Since

$$\int_{U'} \rho\nabla\rho f_1 \nabla f_1 = \frac{1}{4}\int_{U'} \nabla\rho^2 \nabla f_1^2 = -\frac{1}{4}\int_{U'} \Delta\rho^2 f_1^2,$$

and $\nabla\rho, \Delta\rho = 0$ on U, we have

$$\int_{U'} \left(-\rho\Delta\rho f_1^2 - 2\rho\nabla\rho f_1\nabla f_1\right) \leq \frac{1}{\alpha^{2\epsilon}} \int_{U'-U} f_1^2 \leq \frac{\beta}{\alpha^{2\epsilon}} \leq \alpha^{2/3-3\epsilon}.$$

Noting that

$$\int_{U'} \rho^2 f_1^2 \geq \int_U \rho^2 f_1^2 = 1 - \beta \geq 1 - \alpha^{2/3-\epsilon},$$

we then have

$$\lambda_1(U') \leq \lambda_1 + \frac{\alpha^{2/3-3\epsilon}}{1-\alpha^{2/3-\epsilon}} \leq \lambda_1 + \alpha^{2/3-3\epsilon}. \tag{2.12}$$

This gives the required estimate for the first eigenvalue.

2.3.2. *Estimate for* $\lambda_2(U')$. Since ρf_2 is not à priori orthogonal to the first eigenfunction for U', we must modify it to use the Rayleigh quotient (2.3) to estimate $\lambda_2(U')$. Since ρf_1 is not orthogonal to the first eigenfunction for U' because both are positive, there is some $a \in \mathbb{R}$ such that $\rho f_2 + a\rho f_1$ is orthogonal to the first eigenfunction for U'. We may then use $\rho f_2 + a\rho f_1$ as a test function for the Rayleigh quotient on U'. Integrating by parts,

$$\int_{U'} |\nabla \rho f_i|^2 = -\int_{U'} \rho f_i \Delta(\rho f_i) = \lambda_i \int_{U'} \rho^2 f_i^2 - \frac{1}{2}\int_{U'} \nabla \rho^2 \nabla f_i^2 - \int_{U'} \rho \Delta \rho f_i^2. \tag{2.13}$$

We estimate

$$\left| \int_{U'} |\nabla(\rho f_i)|^2 - \lambda_i \int_{U'} \rho^2 f_i^2 \right| \leq \frac{1}{2} \left| \int_{U'} \Delta \rho^2 f_i^2 \right| + \int_{U'} |\rho \Delta \rho| f_i^2,$$

$$\leq \alpha^{-2\epsilon} \int_{U'-U} f_i^2 \leq \alpha^{-2\epsilon} \int_V f_i^2 \leq \alpha^{2/3-3\epsilon}, \tag{2.14}$$

since $\Delta\rho$ and $\nabla\rho$ vanish identically on U. We compute that $\int_{U'} \nabla(\rho f_1)\nabla(\rho f_2) =$

$$-\int_{U'} \rho f_1 \Delta(\rho f_2) = \lambda_2 \int_{U'} \rho^2 f_1 f_2 - 2\int_{U'} \rho \nabla \rho f_1 \nabla f_2 - \int_{U'} \rho f_1 \Delta \rho f_2 \tag{2.15}$$

and $\int_{U'} \nabla(\rho f_1)\nabla(\rho f_2) =$

$$-\int_{U'} \rho f_2 \Delta(\rho f_1) = \lambda_1 \int_{U'} \rho^2 f_1 f_2 - 2\int_{U'} \rho \nabla \rho f_2 \nabla f_1 - \int_{U'} \rho f_2 \Delta \rho f_1. \tag{2.16}$$

This gives the inequality

$$\left| 2\int_{U'} \nabla(\rho f_1)\nabla(\rho f_2) - (\lambda_1 + \lambda_2) \int_{U'} \rho^2 f_1 f_2 \right| \leq \int_{U'-U} \frac{|f_1 f_2|}{\alpha^{2\epsilon}} + 2\left| \int_{U'-U} \rho \nabla \rho \nabla (f_1 f_2) \right|$$

$$\leq \alpha^{-2\epsilon} \left(\int_V |f_1 f_2| + 2\int_V |f_1 f_2| \right) \leq \alpha^{2/3-3\epsilon}, \tag{2.17}$$

which follows from integration by parts and the Schwarz inequality. Expanding,

$$\int_{U'} |\nabla(\rho f_2 + a\rho f_1)|^2 - \lambda_2 \int_{U'} (\rho f_2 + a\rho f_1)^2 = I + II + III,$$

where

$$I = \int_{U'} |\nabla \rho f_2|^2 - \lambda_2 \int_{U'} \rho^2 f_2^2, \quad II = a^2 \left(\int_{U'} |\nabla \rho f_1|^2 - \lambda_2 \int_{U'} \rho^2 f_1^2 \right),$$

and

$$III = \int_{U'} 2a(\nabla \rho f_1)(\nabla \rho f_2) - \lambda_2 \int_{U'} 2a\rho^2 f_1 f_2.$$

By (2.14),

$$I \leq \alpha^{2/3-3\epsilon}. \tag{2.18}$$

To estimate II, we note that $\lambda_1 \leq \lambda_2$ so that

$$II \leq a^2 \left(\int_{U'} |\nabla \rho f_1|^2 - \lambda_1 \int_{U'} \rho^2 f_1^2 \right) \leq a^2 \alpha^{2/3-3\epsilon}, \tag{2.19}$$

which follows from (2.14).

Note that by the orthogonality of f_1 and f_2 and the Schwarz inequality,

$$\left|\int_{U'} \rho^2 f_1 f_2\right| = \left|\int_{ABE} (1-\rho^2) f_1 f_2\right| \le \left|\int_V f_1 f_2\right| \le \alpha^{2/3-\epsilon}. \tag{2.20}$$

By (2.17), (2.20), and adding and subtracting $\lambda_1 \int_{U'} a\rho^2 f_1 f_2$, we estimate III,

$$III \le |a| \left|2\int_{U'} (\nabla \rho f_1)(\nabla \rho f_2) - (\lambda_1+\lambda_2)\int_{U'} \rho^2 f_1 f_2\right| + |a|(\lambda_2-\lambda_1)\left|\int_{U'} \rho^2 f_1 f_2\right|$$

$$\le |a|(\alpha^{2/3-3\epsilon} + (\lambda_2-\lambda_1)\alpha^{2/3-\epsilon}). \tag{2.21}$$

By (2.20), and since $\int_{U'} f_i^2 \ge 1-\alpha^{2/3-\epsilon}$,

$$\int_{U'} (\rho f_2 + a\rho f_1)^2 \ge (1+a^2)(1-\alpha^{2/3-\epsilon}) - 2|a|\alpha^{2/3-3\epsilon}. \tag{2.22}$$

We now estimate the Rayleigh quotient for $\rho f_2 + a\rho f_1$ using (2.18), (2.19), (2.21), and (2.22)

$$\lambda_2(U') \le \lambda_2 + \frac{\alpha^{2/3-3\epsilon} + a^2\alpha^{2/3-3\epsilon} + |a|\alpha^{2/3-3\epsilon} + |a|(\lambda_2-\lambda_1)\alpha^{2/3-\epsilon}}{(1+a^2)(1-\alpha^{2/3-\epsilon}) - 2|a|\alpha^{2/3-3\epsilon}},$$

which shows that

$$\lambda_2(U') \le \lambda_2 + \alpha^{2/3-3\epsilon} + (\lambda_2-\lambda_1)\alpha^{2/3-\epsilon}.$$

2.3.3. *Gap estimate.* Using our estimates for $\lambda_i(U')$,

$$\lambda_2 - \lambda_1 \ge \lambda_2(U') - \lambda_1(U') - \alpha^{2/3-3\epsilon} + (\lambda_2-\lambda_1)\alpha^{2/3-\epsilon} - \alpha^{2/3-3\epsilon}.$$

We then have

$$(\lambda_2-\lambda_1) \ge \frac{1}{1-\alpha^{2/3-\epsilon}}(\lambda_2(U') - \lambda_1(U')) - \alpha^{2/3-3\epsilon},$$

which shows that

$$\lambda_2 - \lambda_1 \ge (\lambda_2(U') - \lambda_1(U')) - O(\alpha^{2/3-3\epsilon}).$$

By the main theorem of [25], since the diameter of U' is at most $4\alpha^\epsilon$,

$$\lambda_2(U') - \lambda_1(U') \ge \frac{\pi^2}{64\alpha^{2\epsilon}},$$

which shows that

$$\lambda_2 - \lambda_1 \ge C\alpha^{-2\epsilon}$$

and is therefore unbounded as $\alpha \to 0$. We have shown that for any triangle with one or two small angles, $\xi(T)$ becomes unbounded as the small angles collapse. Therefore, any sequence of triangles for which $\xi(T)$ is bounded above by a constant C cannot contain a collapsing subsequence. Since it must contain a subsequence of triangles T_n for which $\xi(T_n) \to C$, it must therefore contain a subsequence converging to a triangle T for which $\xi(T) = C$. This completes the proof of the compactness theorem. □

Remark: The numerical methods of [20] use linear combinations of Bessel functions to accurately approximate the Dirichlet eigenvalues of polygonal domains. One may use such methods together with our compactness theorem to numerically prove a lower bound for the gap function on all triangular domains.

3. Fundamental gap on polygonal domains

Some readers may find our compactness result for triangles counterintuitive because of the familiar example of collapsing rectangles.

3.1. The fundamental gap on quadrilateral domains. For a rectangle, R, one may compute the Dirichlet or Neumann eigenvalues and eigenfunctions using separation of variables. For a rectangle with side lengths a, b where we assume $b \leq a$, the first two Dirichlet eigenvlaues are

$$\lambda_1 = \pi^2 \left(\frac{1}{b^2} + \frac{1}{a^2} \right), \quad \lambda_2 = \pi^2 \left(\frac{1}{b^2} + \frac{4}{a^2} \right).$$

The fundamental gap is then

$$\xi(R) = d^2(\lambda_2 - \lambda_1) = \frac{(a^2+b^2)3\pi^2}{a^2}.$$

The gap conjecture becomes sharp when the rectangle collapses as $b \to 0$, and we note that the fundamental gap for the interval $[0, a]$ is

$$\xi([0, a]) = 3\pi^2.$$

This is intuitive since rectangles collapse uniformly to the segment; consider the domain the fixed square $[0, 1] \times [0, 1]$ with coordinates (x, t) where $t = by$. The Laplacian in coordinates (x, t) is related to the Laplacian on the rectangle $[0, 1] \times [0, b]$ with coordinates (x, y) by

$$-\partial_x^2 - \partial_y^2 = -\partial_x^2 - b^2 \partial_t^2.$$

As $b \to 0$, the operator converges in some sense to the operator $-\partial_x^2$ on the interval, and the gap converges to the gap on the interval $[0, 1]$.

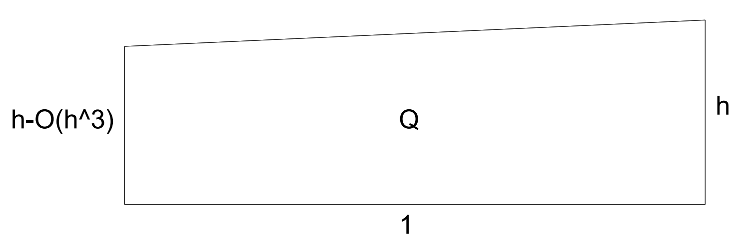

Figure 6. Collapsing quadrilateral with bounded gap.

For triangles, the collapse is not uniform; points opposite the longest side collapse at different rates. Points near the vertex opposite the longest side collapse "more slowly" in some sense than points near the smallest angle. For general polygonal domains, there is a subtle relationship between uniformity of collapse and the behavior of the gap: the examples in Figure 6 and Figure 7 are both approximately rectangular yet in the first example the gap remains bounded as the quadrilateral collapses while in the second example the gap becomes unbounded. This follows from our second theorem.

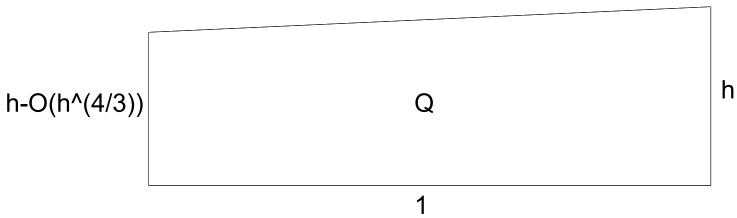

FIGURE 7. Collapsing quadrilateral with unbounded gap.

3.2. **Proof of Theorem 2: bounded gap.** The first case follows almost immediately from domain monotonicity estimates. For ease of notation, we drop the subscript n. The rectangle $R = [0, A] \times [0, B]$. Since Q is collapsing to $[0, 1]$, we must have $B \approx h$ and $A \to 1$. The rectangle $r = [0, a] \times [0, b]$. By the variational principle,

$$\xi(Q) \leq \lambda_2(r) - \lambda_1(R) = \pi^2 \left(\frac{1}{b^2} - \frac{1}{B^2} + \frac{4}{a^2} - \frac{1}{A^2} \right).$$

Since the area of $R \approx h$, by the hypothesis the area of $r \approx h + O(h^3)$. Since $a \leq A$, and $b \leq B$, it follows that $a \to 1$. We calculate

$$\xi(Q) \leq \pi^2 \left(\frac{B^2 - b^2}{(bB)^2} + \frac{4}{a^2} - \frac{1}{A^2} \right).$$

The last two terms are bounded since $a, A \to 1$. Since

$$\frac{B^2 - b^2}{(bB)^2} \sim \frac{hO(h^3)}{h^4}$$

is bounded as $h \to 0$, we see that ξ is bounded as $n \to \infty$.

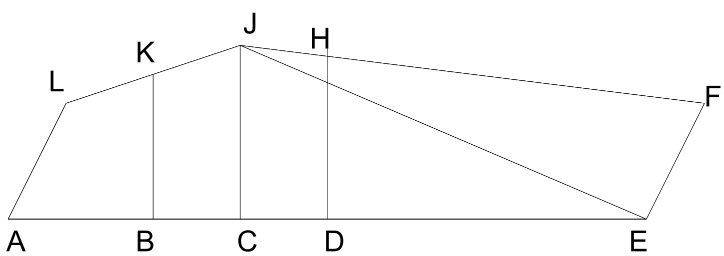

FIGURE 8. Schematic diagram of collapsing polygon with unbounded gap.

3.3. **Proof of Theorem 2: unbounded gap.** We generalize our estimates for arbitrary collapsing triangles. Let $U' \supset U$ be a convex inscribed polygon so that one side Γ of U' satisfies $S \supset \Gamma \supset \Sigma$ and $|\Gamma - \Sigma| = h^\delta$, where $\delta \in (0, 1)$ will be specified later. We can define such an inscribed polygon since the diameter of $U \to 0$, and the length of the longest side of $Q \to 1$. Note that with these hypotheses the diameter of $U' \to 0$ as $h \to 0$. Let f_i be the eigenfunction for Q with eigenvalue λ_i, for $i = 1, 2$. By convexity, Q contains an inscribed right triangle T of height $h = ht(U)$

and base at least $|S|/2$, where $|S|$ is the length of the longest side of Q. Scale Q so that the base of T is one; no generality is lost in showing ξ is unbounded. For illustration, in Figure 8, $Q = AEFJL$, $T = JCE$, $h = |JC|$, and $U = JKBDH$. Assume f_i are normalized so that

$$\int_Q f_i^2 = 1, \text{ and let } \int_V f_i^2 = \beta.$$

By the one dimensional Poincaré inequality,

$$\frac{\int_V |\nabla f_i|^2}{\int_V f_i^2} \geq \frac{\pi^2}{(ht(V))^2},$$

and

$$\frac{\int_U |\nabla f_i|^2}{\int_U f_i^2} \geq \frac{\pi^2}{(ht(U))^2}, \quad \frac{\int_{U'} |\nabla f_i|^2}{\int_{U'} f_i^2} \geq \frac{\pi^2}{(ht(U))^2},$$

since U and U' have the same height. For $h \approx 0$, the measure of the smallest angle of T is approximately $h = ht(U)$. By domain monotonicity,

$$\lambda_i(Q) \leq \lambda_2(T) = \frac{\pi^2}{(ht(U))^2} + \frac{c}{(ht(U))^{4/3}} + O(ht(U)^{-1}).$$

Estimating β as we did for triangles,

$$\frac{\beta\pi^2}{(ht(V))^2} + \frac{\pi^2(1-\beta)}{(ht(U))^2} \leq \int_V |\nabla f_i|^2 + \int_U |\nabla f_i|^2 = \lambda_i = \lambda_2(T) \leq \frac{\pi^2}{(ht(U))^2} + \frac{c}{(ht(U))^{4/3}}.$$

This gives the following estimate for β,

$$\beta \leq \frac{ht(U)^{2/3} ht(V)^2}{ht(U)^2 - ht(V)^2}. \tag{3.1}$$

We have dropped the constant factor. By considering the leading asymptotic behavior as $h \to 0$, we have the estimate

$$\beta \leq h^{5/3-x}. \tag{3.2}$$

Let

$$\epsilon = \frac{5}{3} - x > 0. \tag{3.3}$$

3.3.1. *Estimates for λ_1.* We prove an estimate of the form

$$\lambda_1(U') \leq \lambda_1(Q) + h^y,$$

for some fixed $y > 0$. Define the cut-off function ρ as in (2.9) so that,

$$|\nabla \rho| \leq h^{-\delta}, \quad |\Delta \rho| \leq h^{-2\delta}.$$

Using the test function ρf_1 in the Rayleigh quotient for U', by (2.13),

$$\lambda_1(U') \leq \lambda_1(Q) + \frac{\beta}{h^{2\delta}(1-\beta)} \leq \lambda_1(Q) + h^{\epsilon - 2\delta}. \tag{3.4}$$

We again absorb all constant factors multiplying positive powers of h into a constant factor of 1, since this does not change our limiting estimates for $h \to 0$.

3.3.2. *Estimates for* λ_2. We now require an estimate of the form

$$\lambda_2(U') \leq \lambda_2(Q) + h^y + (\lambda_2(Q) - \lambda_1(Q))h^{y'}$$

for some $y, y' > 0$. Since the function ρf_2 is not à priori admissible as a Rayleigh quotient test function for $\lambda_2(U')$ we again modify it to make it orthogonal to the first eigenfunction on U'. So, we consider the test function

$$\rho f_2 + a\rho f_1.$$

Using (2.13), we estimate

$$\left| \int_{U'} |\nabla(\rho f_i)|^2 - \lambda_i \int_{U'} \rho^2 f_i^2 \right| \leq \frac{1}{2} \left| \int_{U'} \Delta\rho^2 f_i^2 \right| + \int_{U'} |\rho\Delta\rho| f_i^2,$$

$$\leq h^{-2\delta} \int_{U'-U} f_i^2 \leq h^{-2\delta} \int_V f_i^2 \leq h^{\epsilon - 2\delta}, \tag{3.5}$$

since $\Delta\rho$ and $\nabla\rho$ vanish identically on U. By (2.15) and (2.16),

$$\left| 2\int_{U'} \nabla(\rho f_1)\nabla(\rho f_2) - (\lambda_1 + \lambda_2) \int_{U'} \rho^2 f_1 f_2 \right| \leq \int_{U'-U} \frac{|f_1 f_2|}{h^{2\delta}} + 2 \left| \int_{U'-U} \rho\nabla\rho\nabla(f_1 f_2) \right|$$

$$\leq h^{-2\delta} \left(\int_V |f_1 f_2| + 2\int_V |f_1 f_2| \right) \leq h^{\epsilon - 2\delta}, \tag{3.6}$$

which follows from integration by parts and the Schwarz inequality. Expanding,

$$\int_{U'} |\nabla(\rho f_2 + a\rho f_1)|^2 - \lambda_2 \int_{U'} (\rho f_2 + a\rho f_1)^2 = I + II + III,$$

where

$$I = \int_{U'} |\nabla\rho f_2|^2 - \lambda_2 \int_{U'} \rho^2 f_2^2, \quad II = a^2 \left(\int_{U'} |\nabla\rho f_1|^2 - \lambda_2 \int_{U'} \rho^2 f_1^2 \right)$$

and

$$III = \int_{U'} 2a(\nabla\rho f_1)(\nabla\rho f_2) - \lambda_2 \int_{U'} 2a\rho^2 f_1 f_2.$$

By our calculations for triangles, our estimate for β, and the estimates (3.5) and (3.6),

$$I \leq h^{\epsilon - 2\delta}, \tag{3.7}$$

$$II \leq a^2 h^{\epsilon - 2\delta}, \text{ and} \tag{3.8}$$

$$III \leq |a|(h^{\epsilon - 2\delta} + (\lambda_2(Q) - \lambda_1(Q))h^{\epsilon - 2\delta}). \tag{3.9}$$

Moreover,

$$\int_{U'} (\rho f_2 + a\rho f_1)^2 \geq (1 + a^2)(1 - h^\epsilon) - 2|a|h^{\epsilon - 2\delta}. \tag{3.10}$$

Using these estimates, we estimate the Rayleigh quotient for $\rho f_2 + a\rho f_1$,

$$\lambda_2(U') \leq \lambda_2(Q) + \frac{h^{\epsilon-2\delta} + a^2 h^{\epsilon-2\delta} + |a|(h^{\epsilon-2\delta} + (\lambda_2(Q) - \lambda_1(Q))h^{\epsilon-2\delta})}{(1 + a^2)(1 - h^\epsilon) - 2|a|h^{\epsilon-2\delta}}. \tag{3.11}$$

By considering the behavior as $h \to 0$,

$$\lambda_2(U') \leq \lambda_2(Q) + h^{\epsilon - 2\delta} + h^{\epsilon - 2\delta}(\lambda_2(Q) - \lambda_1(Q)).$$

3.3.3. *Gap estimate.* Using our estimates for $\lambda_i(U')$,

$$\lambda_2(Q) - \lambda_1(Q) \geq \lambda_2(U') - \lambda_1(U') - h^{\epsilon-2\delta} + (\lambda_2(Q) - \lambda_1(Q))h^{\epsilon-2\delta},$$

which shows that

$$\lambda_2(Q) - \lambda_1(Q) \geq \frac{\lambda_2(U') - \lambda_1(U')}{1 - h^{\epsilon-2\delta}} - h^{\epsilon-2\delta}. \tag{3.12}$$

Since we are free to choose any $\delta > 0$, we may choose $\delta \in (0, \epsilon/2)$. The diameter of U' is therefore vanishing as $h \to 0$, so (3.12) and the main theorem of [25] imply

$$\lambda_2(Q) - \lambda_1(Q) \to \infty, \quad \text{as } h \to 0.$$

□

Remark: This technical theorem shows that the gap function is sensitive to the rate at which boundary points converge to the longest side. If the height (1.1) of the polygon is h, and if all boundary points are collapsing at the same rate with an error controlled by $O(h^3)$, then the gap remains bounded. However, if the rate of collapse varies by $O(h^x)$ for some $x < 5/3$, then the gap becomes unbounded. For $5/3 \leq x < 3$, our methods do not give information on the gap behavior, however for a generic polygonal domain, we expect that the rate of collapse varies by at least $O(h^{5/3-\epsilon})$. One may conclude from Theorem 2 that generic polygonal domains which collapse to the interval have unbounded gap. This is consistent with [3] and [12]; in particular, see the remarks after Corollary 2.1 of [3].

4. Concluding remarks

We have analyzed the behavior of the gap on the degenerate boundary of the moduli space of triangles and generalized our methods to polygonal domains. Supported by numerical estimates, we conjecture that the gap minimizer for triangular domains is the equilateral triangle. Further analysis of the behavior of ξ on the interior of the moduli space and its non-degenerate boundaries is a reasonable approach to prove the gap conjecture for triangular domains; this is an open problem. For convex polygonal domains of more than three sides, our examples indicate that generically one expects the gap to become unbounded in the moduli space of convex m-gons approaching the degenerate boundaries. Some reasonable open questions are the following. Is it possible to determine necessary and sufficient conditions under which the gap remains bounded as convex polygonal domains collapse to the interval? Among non-collapsing domains, what is the gap minimizer, or does it exist? On the moduli space of convex m-gons, may the gap function have interior extrema, and what are its regularity properties? We hope to inspire readers to attack some of the many open spectral problems, and that our work is a useful contribution to understanding the fundamental gap.

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